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LITTLEWOOD'S MULTIPLE FORMULA FOR SPIN CHARACTERS OF SYMMETRIC GROUPS

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Introduction

This paper deals with some character values of the symmetric group S_n as well as its double cover $\tilde{S}_n$.

Let $\chi^\lambda(\rho)$ be the irreducible character of S_n, indexed by the partition λ and evaluated at the conjugacy class ρ. Comparing the character tables of S_2 and S_4, one observes that

$$\chi^{(4)}(2\rho) = \chi^{(2)}(\rho)$$
$$\chi^{(2^2)}(2\rho) = \chi^{(2)}(\rho) + \chi^{(1^2)}(\rho)$$

for $\rho = (2)$, $2\rho = (4)$ and $\rho = (1^2)$, $2\rho = (2^2)$. A number of such observations lead to what we call *Littlewood's multiple formula* (Theorem 1.1). This formula appears in Littlewood's book [**2**]. We include a proof that is based on an 'inflation' of the variables in a Schur function. This is different from one given in [**2**], and we claim that it is more complete than the one given there.

Our main objective is to obtain the spin character version of Littlewood's multiple formula (Theorem 2.3). Let $\zeta^\lambda(\rho)$ be the irreducible *negative* character of $\tilde{S}_n$ (cf. [**1**]), indexed by the strict partition λ and evaluated at the conjugacy class ρ. One finds character tables $(\zeta^\lambda(\rho))$ in [**1**] for $n \leqslant 14$. This time we evidently see that

$$\zeta^{3\lambda}(3\rho) = \zeta^\lambda(\rho)$$

for $\lambda = (4), (3, 1)$ and $\rho = (3, 1), (1^4)$. The proof of Theorem 2.3 is achieved in a way that is similar to the case of ordinary characters. Instead of a Schur function, we deal with Schur's P-function, which is defined as a ratio of Pfaffians.

1. *Littlewood's multiple formula*

We first recall the multiple formula for ordinary characters of the symmetric groups that is due to Littlewood [**2**, Chapter 8].

Throughout this section, we fix a positive integer r. Let $\lambda = (\lambda_1, \ldots, \lambda_N)$ be a partition. We always assume that N is at least the length $l(\lambda)$ of λ. An $(r+1)$-tuple of partitions $(\lambda^c, \lambda[0], \ldots, \lambda[r-1])$ is attached to λ ; λ^c is the r-core of λ and the collection $\lambda^* = (\lambda[0], \ldots, \lambda[r-1])$ is the r-quotient of λ (cf. [**4**, p. 12]). The effect of changing N is to permute the $\lambda[k]$ cyclically. Since we will only consider the Littlewood–Richardson coefficients $\mathrm{LR}^{\mu}_{\lambda[0],\ldots,\lambda[r-1]}$, the ambiguity of choosing N can be ignored. Put $r\lambda = (r\lambda_1, \ldots, r\lambda_N)$. We can easily verify that $(r\lambda)^c = \emptyset$ and

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$(r\lambda)[0] = (\lambda_r, \lambda_{2r}, \lambda_{3r}, \ldots)$, $(r\lambda)[k] = (\lambda_{r-k}, \lambda_{2r-k}, \lambda_{3r-k}, \ldots)$ $(1 \leqslant k \leqslant r-1)$. As is well known, the ordinary irreducible characters of the symmetric group S_n are parametrized by $P(n)$, the set of all partitions of n. Let $\chi^\lambda(\rho)$ denote the irreducible character of S_n indexed by λ, evaluated at the conjugacy class determined by the partition ρ.

Theorem 1.1 (Littlewood's multiple formula).

$$\chi^{r\lambda}(r\rho) = \sum_{\nu \in P(n)} \mathrm{LR}^{\nu}_{(r\lambda)[0],(r\lambda)[1],\ldots,(r\lambda)[r-1]} \chi^{\nu}(\rho)$$

for any partitions λ and ρ of n.

Littlewood proved a more general formula in [**3**, pp. 340–342]. However, in order to contrast the formula with the spin character case, we here deal only with this form. In the rest of this section, we give a simple proof of the theorem by using Schur functions.

Put $\boldsymbol{x}_N = (x_1, x_2, \ldots, x_N)$, $\boldsymbol{x}_N^r = (x_1^r, x_2^r, \ldots, x_N^r)$ and $\boldsymbol{x}_{N,r} = (x_1, x_2, \ldots, x_{rN})$ with $x_{kN+i} = \omega^k x_i$ $(0 \leqslant k \leqslant r-1,\ 1 \leqslant i \leqslant N)$, where $\omega = \exp(2\pi\sqrt{-1}/r)$. We call $\boldsymbol{x}_{N,r}$ the *r-inflation* of $\boldsymbol{x}_N$. Let $\delta_N = (N-1, N-2, \ldots, 1, 0)$. Following [**4**, p. 40], we define the Schur function of variables $\boldsymbol{x}_N$, corresponding to the partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)$, by

$$s_\lambda(\boldsymbol{x}_N) = \frac{\det A_{\lambda+\delta_N}(\boldsymbol{x}_N)}{\det A_{\delta_N}(\boldsymbol{x}_N)},$$

where

$$A_\alpha(\boldsymbol{x}_N) = (x_i^{\alpha_j})_{1 \leqslant i,j \leqslant N}$$

for a sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ of non-negative integers.

Theorem 1.2.

$$s_{r\lambda}(\boldsymbol{x}_{N,r}) = \prod_{k=0}^{r-1} s_{(r\lambda)[k]}(\boldsymbol{x}_N^r)$$

for any partition $\lambda = (\lambda_1, \ldots, \lambda_{rN})$.

Proof. For a sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ of non-negative integers, put

$$X_N^{\alpha_j+N-j} = \begin{pmatrix} x_1^{\alpha_j+N-j} \\ x_2^{\alpha_j+N-j} \\ \vdots \\ x_N^{\alpha_j+N-j} \end{pmatrix},$$

the jth column of the matrix $A_{\alpha+\delta_N}(\boldsymbol{x}_N)$. We first compute the numerator of $s_{r\lambda}(\boldsymbol{x}_{N,r})$:

$$A_{r\lambda+\delta_{rN}}(\boldsymbol{x}_{N,r}) = (\omega^{(i-1)(r-j)} X_N^{r\lambda_j+rN-j})_{1 \leqslant i \leqslant r, 1 \leqslant j \leqslant rN}.$$

Let $\tau \in S_{rN}$ be defined by

$$\tau(ir+j) = N(j-1)+(i+1) \qquad 0 \leqslant i \leqslant N-1,\ 1 \leqslant j \leqslant r.$$

Permuting columns of the above matrix according to τ, and setting $p =$

diag $(x_1, x_2, \dots, x_N)$, we obtain

$$\begin{aligned}
&\det A_{r\lambda+\delta_{rN}}(\boldsymbol{x}_{N,r}) \\
&\quad = \operatorname{sgn}(\tau)\det(\omega^{(i-1)(r-j)}p^{r-j}A_{(r\lambda)[r-j]+\delta_N}(\boldsymbol{x}^r_N))_{1\leqslant i,j\leqslant r} \\
&\quad = \operatorname{sgn}(\tau)\det(\omega^{(i-1)(r-j)}1_N)_{1\leqslant i,j\leqslant r}\det(p^{r-j})_{1\leqslant j\leqslant r}\det(A_{(r\lambda)[r-j]+\delta_N}(\boldsymbol{x}^r_N))_{1\leqslant j\leqslant r}.
\end{aligned}$$

Thus we have

$$\det A_{r\lambda+\delta_{rN}}(\boldsymbol{x}_{N,r}) = \operatorname{sgn}(\tau)(\det p)^\alpha c\prod_{k=0}^{r-1}\det(A_{(r\lambda)[r-k]+\delta_N}(\boldsymbol{x}^r_N)), \tag{1}$$

where

$$\begin{cases} \alpha = \displaystyle\sum_{j=1}^{r}(r-j) = \binom{r}{2} \\ c = \det(\omega^{(i-1)(r-j)}E_N). \end{cases}$$

Putting $\lambda = (0, \dots, 0)$, we get

$$\det A_{\delta_{rN}}(\boldsymbol{x}_{N,r}) = \operatorname{sgn}(\tau)p^\alpha c\left(\det A_{\delta_N}(\boldsymbol{x}^r_N)\right)^r. \tag{2}$$

From (1) and (2), we see that

$$\begin{aligned}
s_{r\lambda}(\boldsymbol{x}_{N,r}) &= \frac{\det A_{r\lambda+\delta_{rN}}(\boldsymbol{x}_{N,r})}{\det A_{\delta_{rN}}(\boldsymbol{x}_{N,r})} \\
&= \prod_{k=0}^{r-1}\frac{\det A_{(r\lambda)[k]+\delta_{rN}}(\boldsymbol{x}^r_N)}{\det A_{\delta_N}(\boldsymbol{x}^r_N)} \\
&= \prod_{k=0}^{r-1}s_{(r\lambda)[k]}(\boldsymbol{x}^r_N). \qquad \square
\end{aligned}$$

We remark here that the right-hand side of the expression in Theorem 1.2 can also be expressed as

$$\prod_{k=0}^{r-1}s_{\lambda[k]}(\boldsymbol{x}^r_N) = \sum_{\nu\in P(n)}\mathrm{LR}^\nu_{(r\lambda)[0],(r\lambda)[1],\dots,(r\lambda)[r-1]}s_\nu(\boldsymbol{x}^r_N).$$

In order to translate the above identity of Schur functions into that of irreducible characters of S_n, we need the power sum symmetric functions

$$p_m(\boldsymbol{x}_N) = \sum_{i=1}^{N}x_i^m \qquad m = 1, 2, \dots.$$

For a partition $\rho = (\rho_1, \rho_2, \dots)$, define

$$p_\rho = p_{\rho_1}p_{\rho_2}\cdots.$$

The well known Frobenius formula is

$$s_\lambda(\boldsymbol{x}_N) = \sum_{\rho\in P(n)}z_\rho^{-1}\chi^\lambda(\rho)p_\rho(\boldsymbol{x}_N),$$

where

$$z_\rho = \prod_{i=1}^{n}m_i!\ i^{m_i}$$

for the partition $\rho = (1^{m_1} 2^{m_2} \dots n^{m_n})$ of n. As for the r-inflation of $\boldsymbol{x}_N$, we have the following simple fact.

LEMMA 1.3.

$$p_m(\boldsymbol{x}_{N,r}) = \begin{cases} r p_m(\boldsymbol{x}_N) & m \equiv 0 \pmod{r} \\ 0 & m \not\equiv 0 \pmod{r}. \end{cases}$$

Proof. We easily see that

$$\begin{aligned} p_m(\boldsymbol{x}_{N,r}) &= \sum_{i=1}^{rN} x_i^m \\ &= \sum_{i=1}^{N} \left(\sum_{k=0}^{r-1} (\omega^k x_i)^m \right) = \begin{cases} r \sum_{i=1}^{N} x_i^m & m \equiv 0 \pmod{r} \\ \sum_{i=1}^{N} \dfrac{1-(\omega^m)^r}{1-\omega^m} x_i^m = 0 & m \not\equiv 0 \pmod{r}. \end{cases} \end{aligned}$$ □

Combining this lemma with the Frobenius formula, we see that

$$\begin{aligned} s_{r\lambda}(\boldsymbol{x}_{N,r}) &= \sum_{\rho \in P(rn)} z_\rho^{-1} \chi^{r\lambda}(\rho) p_\rho(\boldsymbol{x}_{N,r}) \\ &= \sum_{\rho \in P(n)} z_{r\rho}^{-1} \chi^{r\lambda}(r\rho) r^{l(\rho)} p_{r\rho}(\boldsymbol{x}_N) \\ &= \sum_{\rho \in P(n)} z_\rho^{-1} \chi^{r\lambda}(r\rho) p_\rho(\boldsymbol{x}_N^r). \end{aligned}$$

Therefore we have

$$\begin{aligned} &\sum_{\rho \in P(n)} z_\rho^{-1} \chi^{r\lambda}(r\rho) p_\rho(\boldsymbol{x}_N^r) \\ &\qquad = \sum_{\nu \in P(n)} \mathrm{LR}^{\nu}_{(r\lambda)[0],(r\lambda)[1],\dots,(r\lambda)[r-1]} \left[\sum_{\rho \in P(n)} z_\rho^{-1} \chi^\nu(\rho) p_\rho(\boldsymbol{x}_N^r) \right] \\ &\qquad = \sum_{\rho \in P(n)} \left(\sum_{\nu \in P(n)} \mathrm{LR}^{\nu}_{(r\lambda)[0],(r\lambda)[1],\dots,(r\lambda)[r-1]} \chi^\nu(\rho) \right) z_\rho^{-1} p_\rho(\boldsymbol{x}_N^r). \end{aligned}$$

Since the p_ρ are linearly independent, we obtain

$$\chi^{r\lambda}(r\rho) = \sum_{\nu \in P(n)} \mathrm{LR}^{\nu}_{(r\lambda)[0],(r\lambda)[1],\dots,(r\lambda)[r-1]} \chi^\nu(\rho),$$

as desired.

The proof of Lemma 1.3 given here can immediately be extended to the case of partition $\mu \in P(rn)$ whose r-core is empty. Littlewood's multiple formula is

$$s_\mu(\boldsymbol{x}_{N,r}) = \varepsilon \prod_{k=0}^{r-1} s_{\mu[k]}(\boldsymbol{x}_N^r),$$

and

$$\chi^{\mu}(r\rho) = \varepsilon \sum_{\nu \in P(n)} \mathrm{LR}^{\nu}_{\mu[0],\mu[1],\ldots,\mu[r-1]} \chi^{\nu}(\rho),$$

where ε is the sign depending only on μ.

2. *Spin characters*

We consider Littlewood's multiple formula for spin characters of symmetric groups. Since the theory of spin characters is Pfaffian by nature, Schur's P-functions play an important role. Here we adopt the definition of P-functions due to Nimmo [**6**, Appendix] (see also [**4**, p. 267]).

A partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ is said to be strict if $\lambda_1 > \ldots > \lambda_l > 0$. The set of strict partitions of n is denoted by $\mathrm{SP}(n)$.

Let

$$A(\boldsymbol{x}_N) = \left(\frac{x_i - x_j}{x_i + x_j} \right)_{1 \leqslant i,j \leqslant N},$$

and

$$D_\lambda(\boldsymbol{x}_N) = (x_i^{\lambda_j})_{1 \leqslant i \leqslant N,\ 1 \leqslant j \leqslant l}$$

for a strict partition $\lambda = (\lambda_1, \ldots, \lambda_l)$. Put

$$A_\lambda(\boldsymbol{x}_N) = \begin{pmatrix} A(\boldsymbol{x}_N) & D_\lambda(\boldsymbol{x}_N) \\ -{}^tD_\lambda(\boldsymbol{x}_N) & 0 \end{pmatrix},$$

which is a skew-symmetric matrix of $N + l$ rows and columns. Define $\mathrm{Pf}_\lambda(\boldsymbol{x}_N)$ to be the Pfaffian of $A_\lambda(\boldsymbol{x}_N)$ if $N + l$ is even, and to be the Pfaffian of $A_\lambda(\boldsymbol{x}_{N+1})$ if $N + l$ is odd, agreeing $x_{N+1} = 0$. When $\lambda = \varnothing$, we have

$$\mathrm{Pf}_\varnothing(\boldsymbol{x}_N) = \begin{cases} \mathrm{Pf}A(\boldsymbol{x}_N) & N \text{ is even} \\ \mathrm{Pf}A(\boldsymbol{x}_{N+1}) & N \text{ is odd.} \end{cases}$$

It is a good exercise to verify that

$$\mathrm{Pf}_\varnothing(\boldsymbol{x}_N) = \prod_{1 \leqslant i < j \leqslant N} \frac{x_i - x_j}{x_i + x_j}. \tag{3}$$

Now define Schur's P-function corresponding to the strict partition λ by

$$P_\lambda(\boldsymbol{x}_N) = \frac{\mathrm{Pf}_\lambda(\boldsymbol{x}_N)}{\mathrm{Pf}_\varnothing(\boldsymbol{x}_N)}.$$

Throughout this section we fix a positive odd integer r. The r-inflation of $\boldsymbol{x}_N$ is defined, as in Section 1, by $\boldsymbol{x}_{N,r} = (x_1, x_2, \ldots, x_{rN})$ with $x_{kN+i} = \omega^k x_i$ $(0 \leqslant k \leqslant r-1$, $1 \leqslant i \leqslant N)$, where $\omega = \exp(2\pi\sqrt{-1}/r)$.

LEMMA 2.1.

$$\mathrm{Pf}_\varnothing(\boldsymbol{x}_{N,r}) = c_r \{\mathrm{Pf}_\varnothing(\boldsymbol{x}_N^r)\}^r,$$

where c_r is a non-zero complex number.

Proof. Put

$$B_k = \left(\frac{x_i - \omega^k x_j}{x_i + \omega^k x_j} \right)_{1 \leqslant i,j \leqslant N} \qquad 0 \leqslant k \leqslant r-1,$$

$$\mathscr{B} = (B_{j-i})_{0 \leqslant i,j \leqslant r-1}.$$

Then we see that $\mathscr{B}$ is a skew-symmetric matrix. We also see that

$$\mathrm{Pf}_{\varnothing}(\boldsymbol{x}_{N,r}) = \mathrm{Pf}(\mathscr{B})$$

if N is even, and

$$\mathrm{Pf}_{\varnothing}(\boldsymbol{x}_{N,r}) = \mathrm{Pf}\begin{pmatrix} \mathscr{B} & {}^t\theta \\ -\theta & 0 \end{pmatrix}$$

if N is odd. Here $\theta = (1, \ldots, 1)$ is a row vector of length r. Put

$$b_k^+ = \prod_{1 \leqslant i < j \leqslant N} \frac{x_i - \omega^k x_j}{x_i + \omega^k x_j}, \quad b_k^- = \prod_{1 \leqslant j < i \leqslant N} \frac{x_i - \omega^k x_j}{x_i + \omega^k x_j},$$

and

$$d_k = \left(\frac{1 - \omega^k}{1 + \omega^k}\right)^N$$

for $0 \leqslant k \leqslant r - 1$. Then we check that

$$b_k^- = (-1)^{N(N-1)/2} b_k^+,$$

and

$$\begin{aligned} \prod_{k=0}^{r-1} b_k^+ &= \prod_{1 \leqslant i < j \leqslant N} \left(\prod_{k=0}^{r-1} \frac{x_i - \omega^k x_j}{x_i + \omega^k x_j} \right) \\ &= \prod_{1 \leqslant i < j \leqslant N} \frac{x_i^r - x_j^r}{x_i^r + x_j^r} \\ &= \mathrm{Pf}_{\varnothing}(\boldsymbol{x}_N^r). \end{aligned}$$

By (3), we have

$$\mathrm{Pf}_{\varnothing}(\boldsymbol{x}_{N,r}) = c_r \left(\prod_{k=0}^{r-1} b_k^+ \right)^r = c_r \{\mathrm{Pf}_{\varnothing}(\boldsymbol{x}_N^r)\}^r,$$

where

$$c_r = \prod_{k=1}^{r-1} d_k^{r-k},$$

which is a non-zero constant. □

THEOREM 2.2.

$$P_{r\lambda}(\boldsymbol{x}_{N,r}) = P_{\lambda}(\boldsymbol{x}_N^r)$$

for any strict partition $\lambda = (\lambda_1, \ldots, \lambda_l)$.

Proof. We give a proof for the case where $N + l$ is even, since the other case can be proved with only a slight modification. Set

$$D = D_{\lambda}(\boldsymbol{x}_N^r).$$

Since $rN + l$ is even, we have

$$\mathrm{Pf}_{r\lambda}(\boldsymbol{x}_{N,r}) = \mathrm{Pf}\begin{pmatrix} \mathscr{B} & {}^t\mathscr{D} \\ -\mathscr{D} & 0 \end{pmatrix}.$$

Here $\mathscr{D} = ({}^tD, \ldots, {}^tD)$ is an $N \times rN$ matrix. Apply the following elementary row-block and column-block operations successively to the above matrix:

(1) Subtract the rth row-block from the first, second, $\ldots,(r-1)$th row-blocks.

(2) Subtract the rth column-block from the first, second, $\ldots,(r-1)$th column-blocks.

(3) Add the rth row-block, multiplied by $1/r$, to the first, second, $\ldots,(r-1)$th row-blocks.

(4) Add the rth column-block, multiplied by $1/r$, to the first, second, $\ldots,(r-1)$th column-blocks.

This series of operations preserves the skew-symmetricity of the matrix and does not change its Pfaffian. Therefore we have, setting $B = 1/r\sum_{k=0}^{r-1} B_k$,

$$\begin{aligned}\mathrm{Pf}_{r\lambda}(\boldsymbol{x}_{N,r}) &= \mathrm{Pf}\begin{pmatrix} A' & 0 & 0 \\ 0 & B & D \\ 0 & -{}^tD & 0 \end{pmatrix} \\ &= \mathrm{Pf}(A')\mathrm{Pf}\begin{pmatrix} B & D \\ -{}^tD & 0 \end{pmatrix}\end{aligned}$$

for some skew-symmetric matrix A' of $r(N-1)$ rows and columns. By a simple calculation, we can show that the (i,j)-entry of B is equal to

$$\frac{1}{r}\sum_{k=0}^{r-1}\frac{x_i-\omega^k x_j}{x_i+\omega^k x_j} = \frac{x_i^r - x_j^r}{x_i^r + x_j^r},$$

and thus we see that $\mathrm{Pf}(B) = \mathrm{Pf}_{\varnothing}(\boldsymbol{x}_N^r)$. Hence we have

$$\mathrm{Pf}\begin{pmatrix} B & D \\ -{}^tD & 0 \end{pmatrix} = \mathrm{Pf}_{\lambda}(\boldsymbol{x}_N^r).$$

By the same elementary operations as above, we see that

$$\begin{aligned}\mathrm{Pf}_{\varnothing}(\boldsymbol{x}_{N,r}) &= \mathrm{Pf}\begin{pmatrix} A' & 0 \\ 0 & B \end{pmatrix} \\ &= \mathrm{Pf}(A')\mathrm{Pf}_{\varnothing}(\boldsymbol{x}_N^r).\end{aligned}$$

Since the left-hand side is equal to $c_r\{\mathrm{Pf}_{\varnothing}(\boldsymbol{x}_N^r)\}^r$, we have

$$\mathrm{Pf}(A') = c_r\{\mathrm{Pf}_{\varnothing}(\boldsymbol{x}_N^r)\}^{r-1}.$$

Consequently we see that

$$\begin{aligned}P_{r\lambda}(\boldsymbol{x}_{N,r}) &= \frac{\mathrm{Pf}_{r\lambda}(\boldsymbol{x}_{N,r})}{\mathrm{Pf}_{\varnothing}(\boldsymbol{x}_{N,r})} \\ &= \frac{c_r\{\mathrm{Pf}_{\varnothing}(\boldsymbol{x}_N^r)\}^{r-1}\mathrm{Pf}_{\lambda}(\boldsymbol{x}_N^r)}{c_r\{\mathrm{Pf}_{\varnothing}(\boldsymbol{x}_N^r)\}^r} \\ &= P_{\lambda}(\boldsymbol{x}_N^r).\end{aligned}$$

□

Let $\tilde{S}_n$ $(n \geqslant 4)$ be the double cover of the symmetric group S_n, which is generated by $t_1,\ldots,t_{n-1}$ and z subject to the relations

$$\begin{cases} z^2 = 1 & \\ zt_i = t_i z,\ t_i^2 = z & 1 \leqslant i \leqslant n-1 \\ (t_i t_{i+1})^3 = z & 1 \leqslant i \leqslant n-2 \\ t_i t_j = z t_j t_i & |i-j| \geqslant 2. \end{cases}$$

The group $\tilde{S}_n$ can be regarded as a central extention

$$1 \longrightarrow Q \longrightarrow \tilde{S}_n \xrightarrow{\theta} S_n \longrightarrow 1,$$

where $Q = \{1, z\} \cong S_2$. The central element z acts on an irreducible representation by ± 1. An irreducible representation of $\tilde{S}_n$ is said to be negative if $z = -1$.

The irreducible negative representations of $\tilde{S}_n$ are parametrized up to associativity by $\mathrm{SP}(n)$ [**1**, Chapter 8]. We denote by ζ^λ the character of the irreducible negative representation corresponding to $\lambda \in \mathrm{SP}(n)$. These spin characters are related to the P-functions by the following Schur formula [**5**, p. 64]:

$$P_\lambda(\boldsymbol{x}_N) = \sum_{\rho \in \mathrm{OP}(n)} 2^{(l(\rho) - l(\lambda) + \epsilon(\lambda))/2} z_\rho^{-1} \zeta^\lambda(\rho) p_\rho(\boldsymbol{x}_N),$$

where $\mathrm{OP}(n)$ denotes the set of partitions of n consisting of odd parts, $\zeta^\lambda(\rho)$ is the value of ζ^λ on the conjugacy class determined by $\rho \in \mathrm{OP}(n)$, and

$$\epsilon(\lambda) = \begin{cases} 0 & n - l(\lambda) \text{ is even} \\ 1 & n - l(\lambda) \text{ is odd.} \end{cases}$$

Using the Schur formula, we can translate Theorem 2.2 into the following multiple formula for spin characters.

THEOREM 2.3. *If r is an odd positive integer, we have*

$$\zeta^{r\lambda}(r\rho) = \zeta^\lambda(\rho)$$

for any $\lambda \in \mathrm{SP}(n)$ and $\rho \in \mathrm{OP}(n)$.

Proof. We have

$$\begin{aligned} P_{r\lambda}(\boldsymbol{x}_{N,r}) &= \sum_{\rho \in \mathrm{OP}(rn)} 2^{(l(\rho) - l(r\lambda) + \epsilon(r\lambda))/2} z_\rho^{-1} \zeta^{r\lambda}(\rho) p_\rho(\boldsymbol{x}_{N,r}) \\ &= \sum_{\rho \in \mathrm{OP}(n)} 2^{(l(r\rho) - l(r\lambda) + \epsilon(r\lambda))/2} z_{r\rho}^{-1} \zeta^{r\lambda}(r\rho) r^{l(\rho)} p_\rho(\boldsymbol{x}_N^r) \\ &= \sum_{\rho \in \mathrm{OP}(n)} 2^{(l(r\rho) - l(r\lambda) + \epsilon(r\lambda))/2} z_\rho^{-1} \zeta^{r\lambda}(r\rho) p_\rho(\boldsymbol{x}_N^r). \end{aligned}$$

The left-hand side is equal to

$$P_\lambda(\boldsymbol{x}_N^r) = \sum_{\rho \in \mathrm{OP}(n)} 2^{(l(\rho) - l(\lambda) + \epsilon(\lambda))/2} z_\rho^{-1} \zeta^\lambda(\rho) p_\rho(\boldsymbol{x}_N^r).$$

Since $l(r\rho) = l(\rho)$, $l(r\lambda) = l(\lambda)$ and $\epsilon(r\lambda) = \epsilon(\lambda)$, we can conclude, comparing the coefficients of $p_\rho(\boldsymbol{x}_N^r)$, that $\zeta^{r\lambda}(r\rho) = \zeta^\lambda(\rho)$. □

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References

1. P. N. HOFFMAN and J. F. HUMPHREYS, *Projective representations of the symmetric groups* (Oxford University Press, 1992).
2. D. E. LITTLEWOOD, *The theory of group characters*, 2nd edn (Oxford University Press, 1950).
3. D. E. LITTLEWOOD, 'Modular representations of symmetric groups', *Proc. Roy. Soc. London Ser.* A 209 (1951) 333–353.

4. I. G. MACDONALD, *Symmetric functions and Hall polynomials*, 2nd edn (Oxford University Press, 1995).
5. A. O. MORRIS, 'The spin representation of the symmetric group', *Proc. London Math. Soc.* (3) 12 (1962) 55–76.
6. J. J. C. NIMMO, 'Hall–Littlewood symmetric functions and the BKP equation', *J. Phys.* A 23 (1990) 751–760.

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